

A Schwinger term in q-deformed $su(2)$ algebra***Kazuo Fujikawa and Harunobu Kubo***Department of Physics, University of Tokyo
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Singapore 119260, Republic of Singapore***Abstract**

An extra term generally appears in the q-deformed $su(2)$ algebra for the deformation parameter $q = \exp 2\pi i\theta$, if one combines the Biedenharn-Macfarlane construction of q-deformed $su(2)$, which is a generalization of Schwinger's construction of conventional $su(2)$, with the representation of the q-deformed oscillator algebra which is manifestly free of negative norm. This extra term introduced by the requirement of positive norm is analogous to the Schwinger term in current algebra. Implications of this extra term on the Bloch electron problem analyzed by Wiegmann and Zabrodin are briefly discussed.

The notion of q-deformed algebra[1], which was originally introduced in connection with the inverse scattering problem and the Yang-Baxter equation[2], is going to be a standard machinery of theoretical physics. For example, the q-deformed $su(2)$ for $q = \exp i\pi P/Q$ with mutually prime integers P and Q found a very interesting physical application to the Bloch electrons in two-dimensional lattice model[3-5]. Also, the q-deformed oscillator algebra, which was introduced by Biedenharn[6] and Macfarlane[7] to construct the q-deformed $su(2)$ in the manner of Schwinger's construction of conventional

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$su(2)$, found an interesting implication on the phase operator problem of the photon[8-9]: The real-positive deformation parameter q or $q = \exp 2\pi i\theta$ with an irrational θ gives rise to the conventional Susskind-Glogower phase operator[10], and $q = \exp 2\pi i\theta$ with a rational θ formally gives rise to the Pegg-Barnett phase operator[11]. The singular nature of the transition from one of these two phase operators to the other by a limiting procedure has been analyzed on the basis of the representation of oscillator algebra[12] which is manifestly free of negative norm[13] and the notion of index[14].

In some of physical applications of q -deformed algebra, the notion of Hilbert space with positive definite norm is crucial. This property of positive norm is not quite transparent in the abstract mathematical formulation of q -deformation. The purpose of the present paper is to study to what extent the q -deformed $su(2)$ with $q = \exp 2i\pi\theta$ is modified if one demands that the representation be manifestly free from negative norm. (For real positive q , we do not find an inevitable modification of algebra on the basis of positivity). Our basic strategy to study this problem is to start with the Biedenharn-Macfarlane construction of $su(2)$ by using the representation of the q -deformed oscillator algebra which is manifestly free of negative norm. By this way, we can use the standard Fock space technique with positive definite norm. It is shown that we generally find an extra term ("Schwinger term") in the q -deformed algebra, though in certain cases of physical interest this extra term identically vanishes.

We start with the oscillator algebra introduced by Hong Yan [12]

$$\begin{aligned}
[a, a^\dagger] &= [N_a + 1] - [N_a] \\
[N_a, a^\dagger] &= a^\dagger \\
[N_a, a] &= -a \\
c &= a^\dagger a - [N_a]
\end{aligned} \tag{1}$$

and another set of oscillator variables b, b^\dagger and N_b . The value of the Casimir operator c is chosen to be identical for these two sets of oscillators: $c = b^\dagger b - [N_b]$. The usual notation of $[X] = \sin(2\pi\theta X)/\sin(2\pi\theta)$ for the deformation parameter $q = \exp 2\pi i\theta$ with $-1/2 < \theta < 1/2$ is used. It is known that the algebra (1) supports the Hopf structure[12][15] but not the q -oscillators employed by Refs.[6,7]. Furthermore the latter q -oscillators suffer from a negative norm problem when $q = \exp 2\pi i\theta$ for generic θ .

The representation of the oscillator algebra (1) free of negative norm is defined by[13]

$$\begin{aligned} |l\rangle_a &= \frac{1}{\sqrt{([l - n_0] + [n_0])!}} (a^\dagger)^l |0\rangle \\ |l\rangle_b &= \frac{1}{\sqrt{([l - n_0] + [n_0])!}} (b^\dagger)^l |0\rangle \end{aligned} \quad (2)$$

with $l = 0, 1, 2, \dots$ and the number n_0 , which characterizes the Casimir operator c , is defined to satisfy

$$c = [n_0] = \frac{\sin 2\pi n_0 \theta}{\sin 2\pi \theta} = \frac{1}{|\sin 2\pi \theta|} \quad (3)$$

for $\theta \neq 0$. We also set $a|0\rangle = b|0\rangle = 0$. We then have

$$\begin{aligned} a|l\rangle_a &= \sqrt{[l - n_0] + [n_0]} |l - 1\rangle_a \\ a^\dagger |l\rangle_a &= \sqrt{[l + 1 - n_0] + [n_0]} |l + 1\rangle_a \\ N_a |l\rangle_a &= (l - n_0) |l\rangle_a \end{aligned} \quad (4)$$

and similarly for $|l\rangle_b$.

It is obvious that $[l] = \sin 2\pi \theta l / \sin 2\pi \theta$ can be negative as well as positive for $\theta \neq 0$. In contrast, for the choice of the Casimir operator in (3), we can confirm

$$\begin{aligned} [l - n_0] + [n_0] &= (-\cos 2\pi \theta l + 1) \frac{\sin 2\pi \theta n_0}{\sin 2\pi \theta} \\ &= \frac{1}{|\sin 2\pi \theta|} (1 - \cos 2\pi \theta l) \geq 0 \end{aligned} \quad (5)$$

and thus the representation (4) is free of negative norm for an irrational θ . We thus have

$$\langle l | l' \rangle_a = \delta_{ll'} \quad (6)$$

if $\langle 0 | 0 \rangle = 1$, and $(a)^\dagger = a^\dagger$; similar relations hold for b operators. For a rational $\theta = M/L$, the representation (2) is truncated at $l = L - 1$ but still free of negative norm. Our choice of $c = [n_0]$ in (3) ensures the absence of negative norm. Conversely, one can confirm that $[l - n_0] + [n_0]$ in (5) is made arbitrarily close to zero for a suitable $l (\neq 0)$ for any given $\theta (\neq 0)$, by noting that θ can be approximated arbitrarily accurately by a rational number (i.e., by a ratio of sufficiently large integers), though this does not necessarily mean that the transition from a rational number to an irrational one is smooth. In this sense, our modification of the representation by the Casimir operator c is *minimal*. This

minimal property becomes important later, since the presence of the “Schwinger term” in q-deformed $su(2)$ to be defined later then suggests the inevitable presence of *some* representations which are inflicted by negative norm, if one sets $c = [n_0] = 0$ there.

On the basis of the representations (2) and (3), we define the Biedenharn-Macfarlane construction of q-deformed $su(2)$ generators by

$$\begin{aligned} S_+ &= a^\dagger b \\ S_- &= b^\dagger a \\ S_3 &= \frac{1}{2}(N_a - N_b) \\ \mathcal{C} &= \frac{1}{2}(N_a + N_b) \end{aligned} \tag{7}$$

where \mathcal{C} stands for the Casimir operator of this algebra. On the basis of this definition one finds

$$\begin{aligned} [S_\pm, S_3] &= \mp S_\pm \\ [S_+, S_-] &= [2S_3] + c\{[N_b + 1] - [N_b] - [N_a + 1] + [N_a]\} \\ &= [2S_3] + 4[n_0] \sin \pi\theta \sin 2\pi\theta [S_3] [\mathcal{C} + \frac{1}{2}] \end{aligned} \tag{8}$$

The last term in (8), which is proportional to the Casimir operator c of the oscillator algebra in (3), gives rise to an extra term in the conventional q-deformed $su(2)$ algebra. This extra term emerges through the use of the q-oscillator (1) which has a positive norm representation. The basic reasoning for the existence of the conventional Schwinger term in current algebra [16] was the energy spectrum bounded from below and the positive norm of the Hilbert space. The present construction of (7) may be regarded as a simplest version of current algebra, and for this reason we tentatively call this extra term in (8) as “Schwinger term”, though a more suitable terminology for it may exist. We note that the modified algebra in (1) with a non-trivial Casimir operator, instead of the oscillator algebra in Refs.[6-7] which is obtained by setting $c = 0$ in (1), is crucial to ensure the absence of the negative norm. One can define (8) for a general value of n_0 , but to ensure the absence of the negative norm one has to choose n_0 as in (3), which implies that n_0 is a function of θ . Because of this property, the Schwinger term does not vanish in general even in the limit $\theta \rightarrow 0$. The limit $\theta \rightarrow 0$ is generally singular as was emphasized in Ref.[13].

A $d = 2j + 1$ dimensional (highest weight) representation of the algebra (8) with a Schwinger term is defined on the oscillator Fock space in (2) by

$$\begin{aligned} S_+|j, m\rangle &= \sqrt{([j + m + 1 - n_0] + [n_0])([j - m - n_0] + [n_0])}|j, m + 1\rangle \\ S_-|j, m\rangle &= \sqrt{([j - m + 1 - n_0] + [n_0])([j + m - n_0] + [n_0])}|j, m - 1\rangle \\ S_3|j, m\rangle &= m|j, m\rangle \end{aligned} \quad (9)$$

where

$$|j, m\rangle = |j + m\rangle_a \otimes |j - m\rangle_b, \quad m = -j, -j + 1, \dots, j \quad (10)$$

with $j = 0, 1/2, 1, 3/2, \dots$, and the orthonormality relation

$$\langle j, m|j', m'\rangle = \delta_{jj'}\delta_{mm'} \quad (11)$$

Note that we can satisfy the basic requirement

$$(S_+)^{\dagger} = S_- \quad (12)$$

for the representation in (9), and the highest weight condition $S_+|j, j\rangle = S_-|j, -j\rangle = 0$.

The $2j + 1$ dimensional highest weight representation of the algebra (8) can also be realized by q -difference equations as

$$\begin{aligned} \tilde{S}_+\psi(z) &= (q - q^{-1})^{-1}z(q^{2j-n_0}\psi(q^{-1}z) - q^{-2j+n_0}\psi(qz)) + z[n_0]\psi(z), \\ \tilde{S}_-\psi(z) &= -(q - q^{-1})^{-1}z^{-1}(q^{n_0}\psi(q^{-1}z) - q^{-n_0}\psi(qz)) + z^{-1}[n_0]\psi(z), \\ q^{\tilde{S}_3}\psi(z) &= q^{-j}\psi(qz), \end{aligned} \quad (13)$$

where $\psi(z)$ is a polynomial of degree $2j$. This representation satisfies the highest weight condition $\tilde{S}_+z^{2j} = 0$ and the lowest weight condition $\tilde{S}_- \cdot 1 = 0$. The representation of (13) for the bases, z^{j+m} , $m = j, j - 1, \dots, -j$, is given by

$$\begin{aligned} \tilde{S}_+z^{j+m} &= ([j - m - n_0] + [n_0])z^{j+m+1} \\ \tilde{S}_-z^{j+m} &= ([j + m - n_0] + [n_0])z^{j+m-1} \\ q^{\tilde{S}_3}z^{j+m} &= q^mz^{j+m} \end{aligned} \quad (14)$$

This representation is related to the standard representation in (9) by a (diagonal) similarity transformation A ; $\tilde{S}_+ = AS_+A^{-1}$ and $\tilde{S}_- = AS_-A^{-1}$. Note that A is not unitary, and $\tilde{S}_+^{\dagger} \neq \tilde{S}_-$.

We now discuss the possible implications of our representation (9). A very specific $d = 2j + 1$ dimensional representation for the value of the deformation parameter $q = e^{2\pi i \theta}$ where

$$\theta = \frac{P}{2Q} = \frac{P}{2(2j+1)} \quad (15)$$

with mutually prime integers P and Q found an interesting application in the Bloch electron problem[3-5]. Note that Q and the dimension of the representation $d = 2j + 1$ are independent in general, but in the present case they are related in a specific way. Our states in (2) are sufficient to support this representation since the states in (2) for the value of θ in (15) form a $(2j+1)$ -dimension space for $P = \text{even}$ and a $2(2j+1)$ -dimensional one for $P = \text{odd}$. For the value of θ in (15), the Schwinger term in (8) becomes by noting $2C + 1 = 2j + 1 - 2n_0$ for a $2j + 1$ dimensional representation,

$$\begin{aligned} 4 \sin \pi \theta \left(\frac{\sin 2\pi \theta}{|\sin 2\pi \theta|} \right) [S_3] \frac{\sin \pi \theta (2j + 1 - 2n_0)}{\sin 2\pi \theta} &= \frac{-2}{\cos \pi \theta} \cos \pi \theta (2j + 1) [S_3] \\ &= \frac{-2}{\cos \pi \theta} \cos \left(\frac{\pi}{2} P \right) [S_3] \end{aligned} \quad (16)$$

where we have used $\sin 2\pi n_0 \theta = \sin 2\pi \theta / |\sin 2\pi \theta|$, $\sin^2 2\pi n_0 \theta = 1$, and $\cos 2\pi n_0 \theta = 0$. (Note that the case $j = 0$ and $P = \text{odd}$ is excluded here due to the constraint $-1/2 < \theta < 1/2$ to define $[X] = \sin(2\pi \theta X) / \sin(2\pi \theta)$ for general X .) The Schwinger term in (16) identically vanishes for $P = \text{odd}$, which is one of the allowed cases in the analysis in Ref.[3] and the case analyzed in great detail in Ref.[4]. For this specific case, the Schwinger term identically vanishes and the conventional representation of q -deformed $su(2)$ becomes free of negative norm. In fact, for $P = \text{odd}$, one can confirm that our representation in (9) with the value of n_0 specified there is precisely re-written as

$$S_+ |j, m\rangle = \sqrt{[j + m + 1][j - m]} |j, m + 1\rangle \quad (17)$$

and $S_- = (S_+)^{\dagger}$. This (17) is the conventional representation with $c = [n_0] = 0$.

On the other hand, for $P = \text{even}$, the Schwinger term does not vanish. This suggests that the conventional representation of q -deformed $su(2)$ with $c = [n_0] = 0$ in (9) generally contains negative norm. This is in fact confirmed by noting that

$$\begin{aligned} [j + m + 1][j - m] &= \frac{1}{\sin^2 2\pi \theta} \sin 2\pi \theta (j + m + 1) \sin 2\pi \theta (j - m) \\ &= \frac{1}{2 \sin^2 \left(\pi \frac{P}{2j+1} \right)} \left\{ \cos \left(\pi P \frac{2m+1}{2j+1} \right) \pm 1 \right\} \end{aligned} \quad (18)$$

for $P = \text{odd}$ and $P = \text{even}$, respectively. The minus sign in (18) holds for $P = \text{even}$, and $[j + m - 1][j - m]$ becomes *non-positive*, which spoils $S_- = (S_+)^{\dagger}$ and induces negative norm into the Fock space if one chooses $c = [n_0] = 0$ in (9); in fact, one has $S_- = -(S_+)^{\dagger}$ for $c = [n_0] = 0$. From this view point, it is seen why the *hermitian* Hamiltonian in Ref.[3], where the case $c = [n_0] = 0$ is considered, is fitted by

$$H = i(q - q^{-1})(S_- \pm S_+) \quad (19)$$

with \pm sign corresponding to $P = \text{odd}(\text{even})$, respectively. To be precise,

$$H = i(q - q^{-1})(\rho(S_-) \pm \rho(S_+))$$

by using the cyclic representation in eq(20) below. A cyclic representation corresponding to (9) is obtained by putting $z = q^k$, ($k = 1, 2, \dots, 2Q$) in (13) for the value of θ in (15). There are $2Q$ bases $\psi_k \equiv \psi(q^k)$ which satisfy $\psi_{k+2Q} = \psi_k$, and we define

$$\begin{aligned} \rho(S_+)\psi_k &= \pm(q - q^{-1})^{-1}(q^{k+1+n_0}\psi_{k+1} - q^{k-1-n_0}\psi_{k-1}) + q^k[n_0]\psi_k \\ \rho(S_-)\psi_k &= (q - q^{-1})^{-1}(q^{-k-n_0}\psi_{k+1} - q^{-k+n_0}\psi_{k-1}) + q^{-k}[n_0]\psi_k \\ q^{\rho(S_3)}\psi_k &= q^{-j}\psi_{k+1} \end{aligned} \quad (20)$$

where \pm sign corresponds to $P = \text{odd}(\text{even})$, respectively. It is confirmed that this cyclic representation $\rho(S)$ satisfies the algebra (9) with a Schwinger term given by (16). In particular, the Schwinger term vanishes for $P = \text{odd}$ if one notes $\cos 2\pi n_0 \theta = 0$. This means that the cyclic representation (20) for $P = \text{odd}$ is equivalent to the conventional one in Ref.[3] with $c = [n_0] = 0$. A physical significance of the representation (20) for $P = \text{even}$ with the Schwinger term in (16) is yet to be seen: Group theoretically, one could use $H = i(q - q^{-1})(S_- + S_+)$ in (19) even for $P = \text{even}$ if $c = [n_0]$ is chosen as in (3).

The Schwinger term in (8) vanishes for $\theta = \frac{P}{2(2j+1)}$ with $P = \text{odd}$, whereas the Schwinger term is required to preserve the positive norm of the Hilbert space for $\theta = \frac{P}{2(2j+1)}$ with $P = \text{even}$ or an irrational θ . This fact might be related to the findings in Ref.[4]; it is shown there that the definition of the case of an irrational θ as a limiting case of $\theta = \frac{P}{2(2j+1)}$ with odd P leads to a singular (not differentiable anywhere) behavior of a certain quantity in the Bloch electron problem.

In conclusion, we generally find a Schwinger term in the q -deformed $su(2)$ algebra for $q = e^{2\pi i\theta}$, if one follows the Biedenharn-Macfarlane construction on the basis of the oscillator algebra representation which is manifestly free of negative norm. Mathematically it is not known at this moment if the modification of the q -deformed $su(2)$ algebra by the Schwinger term preserves the Hopf structure or not, but we believe that it is sensible to impose the positive definite norm on the Fock space and to see its physical implications. At least, our Schwinger term is a simple and reliable indicator of negative norm for the representation with $q = e^{2\pi i\theta}$: If the Schwinger term vanishes for a specific representation, it definitely shows that the corresponding conventional representation with $c = [n_0] = 0$ is free of negative norm. On the other hand, the presence of the Schwinger term shows the existence of *some* representations which are inflicted with negative norm if one sets $c = [n_0] = 0$ in (9).

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